Journal of Computational and Applied Mathematics 41 (1992) 281–300
North-Holland

CAM 1063

Banded matrices with banded inverses

Charles A. Micchelli

Department of Mathematical Sciences, IBM T.J. Watson Research Center, Yorktown Heights, NY 10598, United States

Received 13 November 1990
Revised 9 April 1991

Abstract

Micchelli, C.A., Banded matrices with banded inverses, Journal of Computational and Applied Mathematics 41 (1992) 281–300.

Our concern is with the reconstruction of functions from linear observations which only depend locally on the data. We are especially interested in the case that the data and model for reconstruction are "stationary". This leads us to study the solution of *rectangular* Toeplitz equations which have fewer rows than columns. We can therefore take advantage of excess of unknowns over equations to obtain *banded inverses*, even when the original system is banded.

Keywords: Toeplitz matrices; spline functions; interpolation.

1. Introduction

We begin by reviewing some basic facts pertaining to the solution of the bi-infinite Toeplitz equations

$$y_j = \sum_{k \in \mathbb{Z}} a_{j-k} x_k, \quad j \in \mathbb{Z}. \tag{1.1}$$

Equations of this type arise in the problem of reconstructing a function $f(t)$, $t \in \mathbb{R}$, from its values at integers $\{f(j): j \in \mathbb{Z}\}$. For instance, if f is some linear combination of integer translates of another function ϕ, viz.

$$\sum_{j \in \mathbb{Z}} x_j \phi(x-j), \tag{1.2}$$

then the coefficients $\{x_j: j \in \mathbb{Z}\}$ satisfy (1.1) with $y_j = f(j)$ and $a_j = \phi(j)$, $j \in \mathbb{Z}$. Another example is the problem of DC preserving reconstructing a signal. Here the data is

$$y_j = \int_j^{j+1} f(t)\,\mathrm{d}t, \quad j \in \mathbb{Z}, \tag{1.3}$$

Correspondence to: Dr. C.A. Micchelli, Department of Mathematical Sciences, IBM T.J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598, United States.

and the Toeplitz matrix $A = (a_{i-j}\colon i, j \in \mathbb{Z})$ is given by

$$a_j = \int_j^{j+1} \phi(t)\, \mathrm{d}t, \quad j \in \mathbb{Z}. \tag{1.4}$$

When ϕ is a function of compact support, that is, $\phi(x) = 0$, if $|x| \geqslant n$ for some $n \in \mathbb{Z}_+$, the Toeplitz matrix $(a_{i-j}\colon i, j \in \mathbb{Z})$ is banded, that is, $a_j = 0$, $j < j_0$ and $j > j_1$ for some $j_0, j_1 \in \mathbb{Z}$. In this case, the symbol

$$a(z) = \sum_{j \in \mathbb{Z}} a_j z^j \tag{1.5}$$

is a Laurent *polynomial* and the inverse of (1.1) on $l^\infty(\mathbb{Z})$ = the space of bounded bi-infinite sequences is given by

$$x_j = \sum_{k \in \mathbb{Z}} b_{j-k} y_k, \quad j \in \mathbb{Z}, \tag{1.6}$$

where

$$b(z) = \sum_{j \in \mathbb{Z}} b_j z^j := \frac{1}{a(z)}. \tag{1.7}$$

This equation holds in some annulus $\{z\colon r < |z| < r^{-1}\}$, $r > 1$, free of zeros of $a(z)$, and so the sequence $(b_j\colon j \in \mathbb{Z})$ decays exponentially fast. In particular, the existence of the inverse matrix $B = (b_{i-j}\colon i, j \in \mathbb{Z})$ is equivalent to the nonvanishing of the symbol $a(z)$ on the unit circle.

For the interpolation problem described above, the *fundamental function*

$$L(x) = \sum_{j \in \mathbb{Z}} b_j \phi(x - j) \tag{1.8}$$

satisfies

$$L(j) = \delta_{0j}, \quad j \in \mathbb{Z}, \tag{1.9}$$

and

$$(If)(x) := \sum_{j \in \mathbb{Z}} f(j) L(x - j) \tag{1.10}$$

is the desired reconstruction of f from its value on $\mathbb{Z}$. The exponential decay of the sequence $(b_j\colon j \in \mathbb{Z})$ implies that the fundamental function also decays exponentially fast. However, the matrix $(b_{i-j}\colon i, j \in \mathbb{Z})$ *cannot* be banded except in the most trivial of circumstances. This implies that $L(x)$ will not (generally) be of compact support and hence our interpolant (1.10) at any point $x \in \mathbb{R}$ depends on data far from x. Consequently, the use of (1.10) (or (1.8), for that matter) requires a truncation of the infinite series. Therefore it is desirable to have fundamental functions of compact support for numerical computation. Some spline reconstruction methods of this type are given in [3].

In this paper we study this problem from a general perspective. In Section 2 we focus on general multivariate rectangular Toeplitz matrices and give a criteria for the existence of banded inverses.

Some examples of bivariate interpolation by cube splines are given. We also present in the last section reconstruction methods based on integer translates of ripplets, see [7].

2. General theory

We let $\mathscr{A}$ be the Banach algebra of all complex vectors $x=(x_\alpha)_{\alpha\in\mathbb{Z}^s}$, $\{x_\alpha\colon x\in\mathbb{Z}^s\}\subseteq\mathbb{C}$ in $l^1(\mathbb{Z}^s)$ under convolution. Thus for any $x, y\in l^1(\mathbb{Z}^s)$, $x=(x_\alpha)_{\alpha\in\mathbb{Z}^s}$, $y=(y_\alpha)_{\alpha\in\mathbb{Z}^s}$, $\|x\|_1 := \sum_{\alpha\in\mathbb{Z}^s}|x_\alpha|<\infty$ and

$$(x*y)_\alpha := \sum_{\beta\in\mathbb{Z}^s} x_\beta y_{\alpha-\beta}.$$

The unit in $\mathscr{A}$ is $\delta:=(\delta_\alpha)_{\alpha\in\mathbb{Z}^s}$, where $\delta_\alpha=0$, $\alpha\in\mathbb{Z}^s\backslash\{0\}$ and $\delta_0=1$.

With every $x\in l^1(\mathbb{Z}^s)$ we associate the Laurent series

$$x(z):=\sum_{\alpha\in\mathbb{Z}^s} x_\alpha z^\alpha,\quad z=(z_1,\ldots,z_s)\in\mathscr{D}^s,$$

where $\mathscr{D}^s=\{z\colon z=(z_1,\ldots,z_s),\ |z_i|=1,\ i=1,\ldots,s\}$ is the distinguished boundary of the polydisc in $\mathbb{C}^s$. Thus we have the useful identity

$$(x*y)(z)=x(z)y(z),\quad z\in\mathscr{D}^s. \tag{2.1}$$

According to a lemma of Wiener (cf. [12, p.266]), $x\in\mathscr{A}$ is invertible if and only if

$$x(z)\neq 0,\quad z\in\mathscr{D}^s,$$

and in this case

$$\frac{1}{x(z)}=\sum_{\alpha\in\mathbb{Z}^s}(x^{-1})_\alpha z^\alpha,\quad x^{-1}=(x_\alpha^{-1})_{\alpha\in\mathbb{Z}^s}\in\mathscr{A},$$

provides the inverse of x.

The *support* of an $x\in\mathscr{A}$ is defined to be the set of lattice points for which $x_\alpha\neq 0$, that is,

$$\operatorname{supp} x:=\{\alpha\colon x_\alpha\neq 0,\ \alpha\in\mathbb{Z}^s\}.$$

When x is of finite support, that is, $|\operatorname{supp} x|<\infty$, x^{-1} is almost never of finite support. In fact we have the next lemma.

Lemma 2.1. *Let $x, x^{-1}\in\mathscr{A}$; then both x, x^{-1} have compact support if and only if $x=d\delta^\alpha$ for some $\alpha\in\mathbb{Z}^s$ and $d\in\mathbb{C}\backslash\{0\}$, where*

$$(\delta^\alpha)_\beta:=\begin{cases}0, & \alpha\neq\beta,\\ 1, & \alpha=\beta.\end{cases}$$

Proof. If x, x^{-1} are of finite support, the Laurent polynomials $x(z)$, $(x^{-1})(z)$ do not vanish for $z\in(\mathbb{C}\backslash\{0\})^s$ since $x(z)(x^{-1})(z)=1$. It easily follows, for instance by induction on s, that $x(z)=dz^\alpha$ and $d\in\mathbb{C}\backslash\{0\}$, $z\in(\mathbb{C}\backslash\{0\})^s$. $\square$

This observation leads us into the following situation. Let $\mathscr{A}_\infty$ be the Banach space of all bounded sequences on $\mathbb{Z}^s$ with the sup norm. Suppose that $A=(a_{ij})$, $1\leqslant i\leqslant k$, $1\leqslant j\leqslant l$, is a $k\times l$ matrix, $k\leqslant l$, whose elements are in $\mathscr{A}$. A determines an operator on the space

$$\mathscr{A}_\infty^l:=\{\boldsymbol{x}\colon \boldsymbol{x}=(x_1,\ldots,x_l),\ x_1,\ldots,x_l\in\mathscr{A}_\infty\}$$

of l-tuples of all bounded sequences $\mathscr{A}_\infty$ on $\mathbb{Z}^s$ by

$$(Ax)_i := \sum_{j=1}^{l} a_{ij} * x_j, \quad i = 1, \ldots, k.$$

We associate the $k \times l$ complex matrix (symbol)

$$A(z) = \left(a_{ij}(z)\right), \ 1 \leqslant i \leqslant k, \ 1 \leqslant j \leqslant l, \ z \in \mathscr{D}^s,$$

with the operator A.

Theorem 2.2. (i) *$A : \mathscr{A}_\infty^l \to \mathscr{A}_\infty^k$ is surjective if and only if the matrix $A(z)$ is surjective for all $z \in \mathscr{D}^s$. Moreover in this case, there is a matrix $C = (c_{ij})$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant k$, $c_{ij} \in \mathscr{A}$, such that*

$$AC = I_k,$$

where $I_k = \operatorname{diag}_k(\delta, \ldots, \delta)$.

(ii) *If $|\text{supp } a_{ij}| < \infty$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$ and $k \leqslant l$, then we can choose C so that $|\text{supp } c_{ij}| < \infty$ if and only if $A(z)$ is surjective for all $z \in (\mathbb{C} \setminus \{0\})^s$.*

Proof. We begin with (i) and suppose that A is surjective from $\mathscr{A}_\infty^l$ into $\mathscr{A}_\infty^k$. Then for every $z = e^{i\theta}$, $\theta \in \mathbb{R}^s$, and every $t = (t_1, \ldots, t_k) \in \mathbb{C}^k$ there are elements $x_1, \ldots, x_l \in \mathscr{A}_\infty$ such that

$$t_i \eta(\theta) = \sum_{j=1}^{l} a_{ij} * x_j, \quad 1 \leqslant i \leqslant k,$$

where $(\eta(\theta))_\alpha := z^\alpha = e^{i\alpha \cdot \theta}$. We define forward shift operators on $\mathscr{A}_\infty$ as follows:

$$(E_m x)_\alpha = x_{\alpha + e_m}, \quad 1 \leqslant m \leqslant s,$$

where

$$(e_m)_j := \begin{cases} 0, & m \neq j, \\ 1, & m = j. \end{cases}$$

It follows for $1 \leqslant m \leqslant s$ that $E_m \eta(\theta) = e^{i\theta_m} \eta(\theta)$, $\theta = (\theta_1, \ldots, \theta_s)$, and since $E_k(x * y) = (E_k x) * y = x * E_k y$, we get

$$t_i \eta(\theta) = \sum_{j=1}^{l} a_{ij} * x_j^N, \quad 1 \leqslant j \leqslant k,$$

where

$$x^N := \frac{1}{(2N+1)^s} \sum_{|\mu|_\infty \leqslant N} (E^\mu x) \, e^{-i\mu \cdot \theta},$$

for any nonnegative integer N and $x \in \mathscr{A}_\infty$. Here

$$E^\mu = E^{\mu_1} \cdots E_k^{\mu_k}, \quad \mu = (\mu_1, \ldots, \mu_s) \in \mathbb{Z}^s,$$

and $|\mu|_\infty := \max_{1 \leqslant i \leqslant s} |\mu_i|$. Note that for all $N \in \mathbb{Z}_+$,

$$\left\| x_j^N \right\|_\infty := \sup_{\alpha \in \mathbb{Z}^s} \left| \left(x_j^N \right)_\alpha \right| \leqslant \| x_j \|_\infty, \quad 1 \leqslant j \leqslant l.$$

Thus, through some subsequence,

$$\lim_{N' \to \infty} x_j^{N'} = x_j^\infty, \quad 1 \leqslant j \leqslant l,$$

where $x_1^\infty, \ldots, x_l^\infty \in \mathscr{A}_\infty$. Next, we use the fact that for any $x \in \mathscr{A}_\infty$,

$$E_p x^N = \mathrm{e}^{\mathrm{i}\theta_p} x^N + \mathrm{O}\left(\frac{1}{N}\right), \quad 1 \leqslant p \leqslant s,$$

and so in the limit we have $E_p x^\infty = \mathrm{e}^{\mathrm{i}\theta_p} x^\infty$, $1 \leqslant p \leqslant s$. From this equation it follows that $x_j^\infty = \eta(\theta)(x_j^\infty)_0$, $j = 1, \ldots, l$. But for any $x \in \mathscr{A}$, $x * \eta(\theta) = \eta(\theta) x(\mathrm{e}^{-\mathrm{i}\theta})$, and so we finally have

$$t_i = \sum_{j=1}^{l} a_{ij}(\mathrm{e}^{-\mathrm{i}\theta})(x_j^\infty)_0, \quad 1 \leqslant i \leqslant k,$$

that is, the matrix $A(z)$, $z \in \mathscr{D}^s$, is surjective.

Conversely, suppose $A(z): \mathbb{C}^l \to \mathbb{C}^k$ is surjective for all $z \in \mathscr{D}^s$. Then the $k \times k$ matrix

$$A(z)A^*(z)$$

is invertible because, by the Cauchy–Binet formula,

$$\det A(z)A^*(z) = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant k} \left| A(z)\begin{pmatrix} 1, \ldots, k \\ j_1, \ldots, j_k \end{pmatrix} \right|^2 > 0,$$

for each $z \in \mathscr{D}^s$. Here we use the notation $A(z)\binom{1,\ldots,k}{j_1,\ldots,j_k}$ for the $k \times k$ minor of $A(z)$ corresponding to columns $j_1, \ldots, j_k$. By Wiener's Lemma (cf. [12, p.266]) we have

$$(\det A(z)A^*(z))^{-1} = \sum_{\alpha \in \mathbb{Z}^s} d_\alpha z^\alpha, \quad z \in \mathscr{D}^s,$$

for some $d = (d_\alpha)_{\alpha \in \mathbb{Z}^s} \in \mathscr{A}$.

Hence by Cramer's rule we also have

$$A^*(z)(A(z)A^*(z))^{-1} := C(z) = (c_{ij}(z)),$$

for some $c_{ij} \in \mathscr{A}$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant k$, and by construction

$$A(z)C(z) = I_k(z), \quad z \in \mathscr{D}^s, \tag{2.2}$$

where $I_k(z)$ is the $k \times k$ identity matrix on $\mathbb{C}^k$. We claim that $AC = I_k$, that is,

$$\sum_{r=1}^{l} a_{ir} * c_{rj} = \delta\delta_{ij}, \quad 1 \leqslant i \leqslant k, \ 1 \leqslant j \leqslant k. \tag{2.3}$$

But by formula (2.1) this is equivalent to $A(z)C(z) = I_k(z)$. This proves the first claim.

Now, for the second part of the theorem we suppose we can choose a $C = (c_{ij})$, $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant k$, such that

$$\sum_{r=1}^{l} a_{ir} * c_{rj} = \delta\delta_{ij}, \quad |\operatorname{supp} c_{ij}| \leqslant \infty,$$

$1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$. Then $A(z)C(z) = I_k(z)$ for all $z \in (\mathbb{C} \setminus \{0\})^s$ and so $A(z): \mathbb{C}^l \to \mathbb{C}^k$ is surjective for $z \in (\mathbb{C} \setminus \{0\})^s$. For the converse, we appeal to the following theorem.

Theorem 2.3. *Let $A(z)$, $z\in(\mathbb{C}\setminus\{0\})^s$ be a $k\times l$ matrix of Laurent polynomials such that $k\leqslant l$ and*

$$\left\{z: z\in(\mathbb{C}\setminus\{0\})^s,\ \textit{for all}\ 1\leqslant j_1<\cdots<j_l\leqslant l,\ A(z)\begin{pmatrix}1,\ldots,k\\ j_1,\ldots,j_k\end{pmatrix}=0\right\}=\emptyset. \tag{2.4}$$

Then there exists an $l\times k$ matrix $C(z)$ of Laurent polynomials such that

$$A(z)C(z)=I_k(z),\quad z\in(\mathbb{C}\setminus\{0\})^s. \tag{2.5}$$

This fact is known to follow from the Hilbert Nullstellensatz, cf. [13,15]. The condition (2.4) means that $A(z):\mathbb{C}^l\to\mathbb{C}^k$ is surjective for all $z\in(\mathbb{C}\setminus\{0\})^s$. Also, from the Nullstellensatz it follows that the ideal of Laurent polynomials generated by the polynomials

$$P_{j_1,\ldots,j_k}(z):=A(z)\begin{pmatrix}1,\ldots,k\\ j_1,\ldots,j_k\end{pmatrix},\quad 1\leqslant j_1<\cdots<j_k\leqslant l,$$

is trivial.

As for the proof of Theorem 2.3, we define the $l\times k$ matrix $D_{j_1,\ldots,j_k}$:

$$(D_{j_1,\ldots,j_k})_{\mu\nu}=\delta_{\mu j_\nu},\quad \mu=1,\ldots,l,\ \nu=1,\ldots,k.$$

Then for any Laurent polynomials $q_{j_1,\ldots,j_k}(z)$ the $l\times k$ matrix

$$C(z)=\sum_{1\leqslant j_1<\cdots<j_k\leqslant l} q_{j_1,\ldots,j_k}(z)D_{j_1,\ldots,j_k}(z)\ \operatorname{adj}\big(A(z)D_{j_1,\ldots,j_k}(z)\big) \tag{2.6}$$

satisfies the equation

$$A(z)C(z)=\left(\sum_{1\leqslant j_1<\cdots<j_k\leqslant l} q_{j_1,\ldots,j_k}(z)A(z)\begin{pmatrix}1,\ldots,k\\ j_1,\ldots,j_k\end{pmatrix}\right)I_k(z).$$

We now choose Laurent polynomials $q_{j_1,\ldots,j_k}$ so that

$$\sum_{1\leqslant j_1<\cdots<j_k\leqslant l} q_{j_1,\ldots,j_k}(z)A(z)\begin{pmatrix}1,\ldots,k\\ j_1,\ldots,j_k\end{pmatrix}=1,\quad z\in(\mathbb{C}\setminus\{0\})^s.$$

For this choice the matrix (2.6) satisfies (2.5). □ (Theorem 2.3)

Returning to Theorem 2.2, we see the remainder of the proof uses the matrix $C(z)$ in (2.5) to verify, as in part (i), that $AC=I_k$. □

As a corollary, we obtain a result proved in [9] by a different argument.

Corollary 2.4. *Suppose that $|\operatorname{supp} a_{ij}|<\infty$, $1\leqslant i\leqslant k$, $1\leqslant j\leqslant l$ with $k\leqslant l$. Then the system of difference equations*

$$\sum_{j=1}^{k} a_{ji}*d_j=0,\quad 1\leqslant i\leqslant l, \tag{2.7}$$

has only the zero solution if and only if $A(z):\mathbb{C}^l\to\mathbb{C}^k$ is surjective for all $z\in(\mathbb{C}\setminus\{0\})^s$.

Proof. Suppose $A(z)$ is surjective for all $z \in (\mathbb{C}\setminus\{0\})^s$. Then according to Theorem 2.2 there are elements $c_{rj} \in \mathscr{A}$, $|\operatorname{supp} c_{rj}| < \infty$, $1 \leqslant r \leqslant l$, $1 \leqslant j \leqslant k$, such that

$$\sum_{r=1}^{l} a_{jr} * c_{rm} = \delta\delta_{jm}, \quad 1 \leqslant j,m \leqslant k.$$

Consequently, we have

$$0 = \sum_{r=1}^{l} \sum_{j=1}^{k} a_{jr} * d_j * c_{rm} = \sum_{j=1}^{k} (\delta * d_j)\delta_{jm} = d_m.$$

Conversely, if there is a $z_0 \in (\mathbb{C}\setminus\{0\})^s$ such that $A(z_0)$ is not surjective, then there is a vector $t = (t_1, \ldots, t_k) \in \mathbb{C}^k\setminus\{0\}$ such that

$$\sum_{j=1}^{k} a_{ji}(z_0)t_j = 0, \quad 1 \leqslant i \leqslant l.$$

Set $d_j = \eta(\mathrm{e}^{-\mathrm{i}\theta_0})t_j$, $1 \leqslant j \leqslant k$, where $z_0 = \mathrm{e}^{\mathrm{i}\theta_0}$, for some $\theta_0 \in \mathbb{C}^s$. Then it follows directly that $d_1, \ldots, d_k$ satisfies the difference equation (2.7) above. This proves the result. □

3. Local interpolation

In this section we apply Theorem 2.2 to multivariate interpolation on the regular lattice $\mathbb{Z}^s$. Thus we want to find a fundamental function L,

$$L(\alpha) = \delta_{0\alpha}, \quad \alpha \in \mathbb{Z}^s, \tag{3.1}$$

which means that the linear operator

$$(If)(x) = \sum_{\alpha \in \mathbb{Z}^s} f(\alpha)L(x-\alpha) \tag{3.2}$$

interpolates f at α, viz.

$$(If)(\alpha) = f(\alpha), \quad \alpha \in \mathbb{Z}^s. \tag{3.3}$$

We are interested in fundamental functions constructed as follows:

$$L(x) = \sum_{\alpha \in \mathbb{Z}^s} c_\alpha^1 \phi_1(x-\alpha) + \cdots + \sum_{\alpha \in \mathbb{Z}^s} c_\alpha^m \phi_m(x-\alpha), \quad x \in \mathbb{R}^s, \tag{3.4}$$

where $\phi_1, \ldots, \phi_m$ are functions of compact support.

Theorem 2.2 says there is a fundamental function of the form (3.4) which decays exponentially fast, provided that the Laurent polynomials

$$\Phi^j(z) = \sum_{\alpha \in \mathbb{Z}^s} \phi_j(\alpha)z^\alpha, \quad j = 1, \ldots, m, \tag{3.5}$$

have no common zeros on $\mathscr{D}^s$. In this case the coefficients

$$c_\alpha^j := \frac{1}{(2\pi)^s}\int_{[0,2\pi]^s} \mathrm{e}^{-\mathrm{i}\alpha\cdot\omega} \frac{\overline{\Phi^j(\mathrm{e}^{\mathrm{i}\omega})}}{\sum_{j=1}^{m} |\Phi^j(\mathrm{e}^{\mathrm{i}\omega})|^2}\mathrm{d}\omega, \quad \alpha \in \mathbb{Z}^s,\ j = 1, \ldots, m,$$

give one possible solution of (3.1).

A *compactly supported* fundamental function of the desired form (3.4) (by which we mean that the sets $\{\alpha: c_\alpha^l \neq 0,\ \alpha \in \mathbb{Z}^s\}$, $l = 1,\ldots,m$, are finite) requires that the Laurent polynomials (3.5) have no common zeros in $(\mathbb{C}\setminus\{0\})^s$. In this case, to find the coefficients of L we must solve the algebraic equation

$$1 = \sum_{j=1}^{m} c^j(z)\Phi^j(z), \quad z \in (\mathbb{C}\setminus\{0\})^s, \tag{3.6}$$

for some Laurent polynomials $c^j(z)$, $1 \leqslant j \leqslant m$. These equations can be solved symbolically by using Gröbner bases methods, cf. [1].

Generally, Gröbner bases computation is used to solve the problem of determining if a polynomial h is in the ideal generated by a given set of polynomials $p_1,\ldots,p_m$ and, if so, to determine polynomials $g_1,\ldots,g_m$ so that $h = p_1g_1 + \cdots + p_mg_m$. We can reduce the solution of (3.6) to this question by setting $\zeta = (\zeta_1,\ldots,\zeta_s)$ and $z = (z_1,\ldots,z_s)$ and introducing the polynomials

$$\tilde{\Phi}^j(z, \zeta) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_s)\in\mathbb{Z}^s} \phi_j(\alpha) \prod_{\alpha_j>0} z_j^{\alpha_j} \prod_{\alpha_j<0} \zeta^{-\alpha_j}$$

on $\mathbb{C}^{2s}$. Then $\tilde{\Phi}^j(z, z^{-1}) = \Phi^j(z)$, $1 \leqslant j \leqslant m$. Moreover, if $\Phi^j(z)$, $1 \leqslant j \leqslant m$, have no common zeros in $(\mathbb{C}\setminus\{0\})^s$, the Nullstellensatz insures that there are polynomials $g_1(z, \zeta),\ldots,g_s(z, \zeta)$, $d_1(z, \zeta),\ldots,d_m(z, \zeta)$ such that

$$1 = \sum_{i=1}^{s} (1 - z_i\zeta_i)g_i(z, \zeta) + \sum_{j=1}^{m} d_j(z, \zeta)\tilde{\Phi}^j(z, \zeta),$$

and so $c^j(z) := d_j(z, z^{-1})$, $1 \leqslant j \leqslant m$, solves (3.6).

A useful method for choosing the functions $\phi_1,\ldots,\phi_m$ is to start with *one* continuous function ϕ of compact support and express the fundamental function in the form

$$L(x) = \sum_{\alpha\in\mathbb{Z}^s} d_\alpha\phi(2x - \alpha), \quad x \in \mathbb{R}^s. \tag{3.7}$$

This has the previous form with the identification $m = 2^s$,

$$\phi^e(x) := \phi(2x - e), \quad e \in E_s := \text{extreme points of } [0, 1]^s,$$

$$c_\beta^e := d_{2\beta+e}, \quad e \in E_s, \ \beta \in \mathbb{Z}^s,$$

and

$$\Phi^e(z) = \sum_{\beta\in\mathbb{Z}^s} \phi(2\beta + e)z^\beta, \quad e \in E_s.$$

When these Laurent polynomials have no common zeros in $(\mathbb{C}\setminus\{0\})^s$, the coefficients of the fundamental function L in (3.7) are given by the algebraic equation

$$\sum_{e\in E_s} c^e(z)\Phi^e(z)z^e = 1, \quad z \in (\mathbb{C}\setminus\{0\})^s. \tag{3.8}$$

Let us rewrite this equation in an equivalent form. First observe that

$$\sum_{e\in E_s}(-1)^{e\cdot e'}z^e\Phi^e(z^2)=\sum_{e\in E_s}\sum_{\beta\in\mathbb{Z}^s}\phi(2\beta+e)z^{2\beta+e}(-1)^{e\cdot e'}$$
$$=\sum_{\gamma\in\mathbb{Z}^s}\phi(\gamma)\big(z(-1)^{e'}\big)^{\gamma}. \tag{3.9}$$

Since $\det((-1)^{e\cdot e'}\colon e'\in E_s)=-2^s\neq 0$, the existence of a solution to (3.8) is equivalent to the requirement that the polynomials

$$\Phi\big(z(-1)^e\big),\quad e\in E_s,$$

where

$$\Phi(z):=\sum_{\gamma\in\mathbb{Z}^s}\phi(\gamma)z^{\gamma},$$

have no common zeros in $(\mathbb{C}\setminus\{0\})^s$. Furthermore, using the fact that

$$\sum_{e'\in E_s}(-1)^{e\cdot e'}=2^s\delta_{0e},\quad e\in E_s,$$

we have

$$\sum_{e\in E_s}d\big(z(-1)^e\big)\Phi\big(z(-1)^e\big)=\sum_{e\in E_s}\sum_{e'\in E_s}\sum_{e''\in E_s}c^{e'}(z^2)z^{e'}\Phi^{e''}(z^2)z^{e''}(-1)^{(e'-e'')\cdot e}$$
$$=2^s\sum_{e\in E_s}c^e(z^2)\Phi^e(z^2)z^{2e}.$$

Thus equation (3.8) is equivalent to the identity

$$2^s=\sum_{e\in E_s}d\big(z(-1)^e\big)\Phi\big(z(-1)^e\big). \tag{3.10}$$

In summary, we have the following theorem.

Theorem 3.1. *Let ϕ be a function of compact support. There exists a fundamental function of the form*

$$L(x)=\sum_{\alpha\in\mathbb{Z}^s}d_\alpha\phi(2x-\alpha),$$

for some sequence $d=(d_\alpha\colon \alpha\in\mathbb{Z}^s)$, of finite support if and only if the Laurent polynomials

$$\Phi\big(z(-1)^e\big),\quad e\in E_s,$$

have no common zeros in $(\mathbb{C}\setminus\{0\})^s$. Moreover, in this case any sequence $(d_\alpha\colon \alpha\in\mathbb{Z}^s)$ satisfying

$$2^s=\sum_{e\in E_s}d\big(z(-1)^e\big)\Phi\big(z(-1)^e\big),\quad z\in(\mathbb{C}\setminus\{0\})^s,$$

provides a fundamental function.

If we allow in (3.7) an integer other than two, it becomes much easier to construct a fundamental function of compact support. Suppose that ϕ is a continuous function of compact support such that there is a $\beta \in$ interior(supp ϕ) with $\beta = p\gamma$, $p \in \mathbb{Z}_+$, $\gamma \in \mathbb{Z}^s$, and for every $\alpha \in \mathbb{Z}^s \setminus \{\gamma\}$, $p\alpha \notin$ interior(supp ϕ). Then $L(x) = \phi(px + \beta)/\phi(\beta)$ is a fundamental function. This method may reduce the accuracy of the interpolation scheme (3.2). For instance, suppose ϕ satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} \phi(x - \alpha) = 1, \quad x \in \mathbb{R}^s,$$

equivalently (by the Poisson summation formula),

$$\hat{\phi}(2\pi\alpha) = \delta_{0\alpha}, \quad \alpha \in \mathbb{Z}^s, \tag{3.11}$$

where

$$\hat{\phi}(\omega) = \int_{\mathbb{R}^s} e^{i\omega \cdot x} \phi(x)\, dx$$

is the Fourier transform of ϕ. But $\sum_{\gamma \in \mathbb{Z}^s} L(x - \gamma)$ may not be one, because

$$\hat{L}(\omega) = p^{-s}\, e^{-i\beta \cdot \omega/p} \frac{\hat{\phi}(\omega/p)}{\phi(\beta)},$$

and thus (3.11) does not necessarily follow for the fundamental function L.

The following result gives some indication of how to maintain accuracy.

Proposition 3.2. *Suppose ϕ is a continuous function of compact support such that*

$$\hat{\phi}(0) = 1, \qquad \hat{\phi}(e\pi) \neq 0, \quad e \in E_s \setminus \{0\}. \tag{3.12}$$

Let L be a fundamental function of the form (3.7) *corresponding to a* $(d_\alpha\colon \alpha \in \mathbb{Z}^s)$ *of finite support satisfying* (3.10). *Then the corresponding interpolation operator I defined by* (3.2) *satisfies*

$$Ip = p, \quad \text{all } p \in \pi_k(\mathbb{R}^s), \tag{3.13}$$

$\pi_k(\mathbb{R}^s)$ = *polynomials of total degree* $\leqslant k$, *if and only if for all* $e \in E_s \setminus \{0\}$,

$$d(1) = 2^s, \qquad (D^\beta d)((-1)^e) = 0, \quad |\beta| \leqslant k, \tag{3.14}$$

and

$$(D^\beta \hat{\phi})(2\pi\alpha) = 0, \quad \alpha \in \mathbb{Z}^s \setminus \{0\},\ |\beta| \leqslant k, \tag{3.15}$$

where

$$|\beta| := |\beta_1| + \cdots + |\beta_s| \leqslant k, \quad \beta = (\beta_1, \ldots, \beta_s),$$

and

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_s^{\beta_s}}.$$

Proof. We let $T(\omega) = d(e^{-i\omega})$ and observe by the chain rule that the condition (3.14) is equivalent to $T(0) = 2^s$ and $(D^\beta T)(\pi e) = 0$ for all $e \in E_s \setminus \{0\}$ and $|\beta| \leqslant k$. We will use (3.14) in this form and first establish the sufficiency of (3.14) and (3.15). In fact, we will show that (3.14) and (3.15) imply that $\hat{L}$ also satisfies (3.15) and $\hat{L}(0) = 1$. This will insure, by the Poisson summation formula, that $I(\pi_k(\mathbb{R}^s)) \subseteq \pi_k(\mathbb{R}^s)$, and so, since If interpolates f on $\mathbb{Z}^s$, (3.13) would follow. Now, Leibnitz's rule implies

$$(\mathrm{D}^\beta \hat{L})(2\pi\alpha) = 2^{-s-|\beta|} \sum_{0 \leqslant \mu \leqslant \beta} \binom{\beta}{\mu} (\mathrm{D}^\mu T)(\pi e)(\mathrm{D}^{\beta-\mu}\hat{\phi})(2\pi\gamma + \pi e), \tag{3.16}$$

where $\alpha = 2\gamma + e$. Here we have used the 2π-periodicity of T. According to (3.14) it follows that $(\mathrm{D}^\beta \hat{L})(2\pi\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \backslash 2\mathbb{Z}^s$ and $|\beta| \leqslant k$. If $\alpha \in 2\mathbb{Z}^s \backslash \{0\}$, then (3.15) and (3.16) with $e = 0$ imply that $(\mathrm{D}^\beta \hat{L})(2\pi\alpha)$ is again zero. Hence, $\hat{L}$ indeed satisfies (3.15) with $\hat{L}(0) = 2^{-s} d(1)\hat{\phi}(0) = 1$.

For the converse, suppose that (3.13) is valid. Then it is well known that $\hat{L}$ satisfies (3.15) (with ϕ replaced by L) and also $\hat{L}(0) = 1$. Thus $d(1) = 2^s$, since $\hat{\phi}(0) = 1$ by assumption. Moreover, for $e \in E_s \backslash \{0\}$ and $|\beta| \leqslant k$,

$$\begin{aligned} 0 = (\mathrm{D}^\beta \hat{L})(2\pi e) &= 2^{-s-|\beta|}(\mathrm{D}^\beta T)(\pi e)\hat{\phi}(\pi e) \\ &\quad + 2^{-s-|\beta|} \sum_{\substack{0 \leqslant \mu \leqslant \beta \\ \mu \neq \beta}} \binom{\beta}{\mu} (\mathrm{D}^\mu T)(\pi e)(\mathrm{D}^{\beta-\mu}\hat{\phi})(\pi e). \end{aligned}$$

Hence, whenever $(\mathrm{D}^\mu T)(\pi e) = 0$, for all $\mu \in \mathbb{Z}^s$ such that $0 \leqslant \mu \leqslant \beta$, $\mu \neq \beta$, it follows that $(\mathrm{D}^\beta T)(\pi e) = 0$. Thus we conclude inductively on $|\beta|$ that (3.14) holds. As for (3.15), we have for $\alpha = 2\gamma \neq 0$ and $|\beta| \leqslant k$,

$$\begin{aligned} 0 &= (\mathrm{D}^\beta \hat{L})(2\pi\alpha) \\ &= 2^{-s-|\beta|} \left\{ (\mathrm{D}^\beta \hat{\phi})(2\pi\gamma) 2^s + \sum_{\substack{0 \leqslant \mu \leqslant \beta \\ \mu \neq 0}} \binom{\beta}{\mu} (\mathrm{D}^\mu T)(0)(\mathrm{D}^{\beta-\mu}\hat{\phi})(2\pi\gamma) \right\}. \end{aligned}$$

Hence, whenever $(\mathrm{D}^\mu \hat{\phi})(2\pi\gamma) = 0$, for all $\mu \in \mathbb{Z}^s$ with $0 \leqslant \mu \leqslant \beta$, $\mu \neq \beta$, then $(\mathrm{D}^\beta \hat{\phi})(2\pi\gamma) = 0$. Therefore, (3.15) also follows inductively on $|\beta|$. □

Remark 3.3. The same proof provides a similar result for coordinate degree polynomials. Also, we remark that Proposition 3.2 shows that when ϕ satisfies (3.12), then L given by (3.7) with $\hat{L}(0) = 1$ satisfies (3.15) (with ϕ replaced by L), if and only if ϕ satisfies (3.15) and $d(z)$ satisfies (3.14).

Next, we discuss a method suggested by results in [3] which is guaranteed to preserve accuracy. Let ϕ be a function of compact support. Suppose that $\mathscr{P}$ is a finite-dimensional subspace of polynomials invariant under an arbitrary shift, that is, $S_y p := p(\cdot + y) \in \mathscr{P}$ for all $y \in \mathbb{R}^s$ whenever $p \in \mathscr{P}$. We also require that $\mathscr{P}$ is an invariant subspace of the linear operator

$$(Tf)(x) = \sum_{\alpha \in \mathbb{Z}^s} f(\alpha)\phi(x - \alpha), \quad x \in \mathbb{R}^s. \tag{3.17}$$

As mentioned in the proof of Proposition 3.2, conditions on the zeros of the Fourier transform of ϕ will insure that this is the case when $\mathscr{P} = \pi_k(\mathbb{R}^s)$. For similar results in the general case, see [2].

Consider the interpolation operator I of the form

$$(If)(x) := \sum_{\alpha \in \mathbb{Z}^s} f(\alpha)L(x-\alpha), \tag{3.18}$$

with a fundamental function of the form

$$L(x) = \sum_{\alpha \in \mathbb{Z}^s} m_\alpha \phi(x - \tfrac{1}{2}\alpha), \tag{3.19}$$

where $(m_\alpha: \alpha \in \mathbb{Z}^s)$ is a sequence of finite support. It follows easily that I also has $\mathscr{P}$ as an invariant subspace since

$$I = \sum_{\beta \in \mathbb{Z}^s} m_\beta S_{-\beta/2} T.$$

Thus we have

$$Ip = p, \quad \text{all } p \in \mathscr{P}.$$

According to Theorem 2.2 a fundamental function of compact support of the type (3.19) exists if and only if the Laurent polynomials

$$q^e(z) := \sum_{\alpha \in \mathbb{Z}^s} \phi(\alpha - \tfrac{1}{2}e) z^\alpha, \quad e \in E_s,$$

have no common zeros in $(\mathbb{C} \setminus \{0\})^s$. Moreover, in this case any sequence $(m_\alpha: \alpha \in \mathbb{Z}^s)$ satisfying the equation

$$\sum_{e \in E_s} m^e(z) q^e(z) = 1, \quad z \in (\mathbb{C} \setminus \{0\})^s,$$

gives a fundamental function where

$$m^e(z) = \sum_{\alpha \in \mathbb{Z}} m_{2\alpha+e} z^\alpha, \quad z \in (\mathbb{C} \setminus \{0\})^s.$$

When ϕ satisfies a refinement equation

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \phi(2x - \alpha), \tag{3.20}$$

for some finite mask $\{a_\alpha: \alpha \in \mathbb{Z}^s\}$ (this equation is extensively studied in [2]), the fundamental function (3.19) has the form (3.7)

$$L(x) = \sum_{\beta \in \mathbb{Z}^s} \left(\sum_{\gamma \in \mathbb{Z}^s} a_\gamma m_{\beta-\gamma} \right) \phi(2x - \beta), \quad x \in \mathbb{R}^s. \tag{3.21}$$

Hence we have the following result.

Theorem 3.4. *Let ϕ be a continuous function of compact support satisfying the refinement equation* (3.20) *with a finite mask. Then there exists an L of the form*

$$L(x) = \sum_{\alpha \in \mathbb{Z}^s} m_\alpha \phi(x - \tfrac{1}{2}\alpha), \quad x \in \mathbb{R}^s, \tag{3.22}$$

satisfying

$$L(\alpha) = \delta_{0\alpha}, \quad \alpha \in \mathbb{Z}^s, \tag{3.23}$$

where $(m_\alpha: \alpha \in \mathbb{Z}^s)$ *is a sequence of finite support if and only if the Laurent polynomials*

$$a\left(z(-1)^e\right)\Phi\left(z(-1)^e\right), \quad e \in E_s,$$

have no common zeros in $(\mathbb{C}\setminus\{0\})^s$. *In this case*, (3.23) *is equivalent to the algebraic identity*

$$2^s = \sum_{e \in E_s} m\left(z(-1)^e\right)a\left(z(-1)^e\right)\Phi\left(z(-1)^e\right), \quad z \in (\mathbb{C}\setminus\{0\})^s. \tag{3.24}$$

Proof. According to formula (3.21) and (3.10) for the fundamental function (3.7) we see that (3.23) is equivalent to (3.24). □

We give several examples of these results using bivariate cube splines. Recall, in general the cube spline $c(x \mid X)$, $x \in \mathbb{R}^s$, where $X = \{x^1, \ldots, x^n\} \subseteq \mathbb{Z}^s\setminus\{0\}$ is the distribution defined by

$$\int_{\mathbb{R}^s} f(x)c(x \mid X)\,\mathrm{d}x = \int_{[0,1]^n} f(Xt)\,\mathrm{d}t, \quad f \in C(\mathbb{R}^s). \tag{3.25}$$

When the vectors X span $\mathbb{R}^s$, $c(\cdot \mid X)$ is a density which is a piecewise polynomial, cf. [4] and references therein for facts about the cube spline. We only need to know here that

$$c(x \mid X) = \sum_{\alpha \in \mathbb{Z}^s} b_\alpha c(2x - \alpha \mid X), \quad x \in \mathbb{R}^s, \tag{3.26}$$

where

$$b(z \mid X) := \sum_{\alpha \in \mathbb{Z}^s} b_\alpha z^\alpha = 2^{-n+s}\prod_{i=1}^{n}(z^{x^i} + 1), \quad z \in (\mathbb{C}\setminus\{0\})^s,$$

a result proved in [5]; also, we set

$$\Phi(z \mid X) := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha \mid X)z^\alpha \quad \text{and} \quad \psi(z \mid X) = b(z \mid X)\Phi(z \mid X).$$

To apply Theorem 3.4 for $\phi = c(\cdot \mid X)$ we need to compute the values of ϕ at all lattice points $\alpha \in \mathbb{Z}^s$. According to the refinement equation (3.26) the values $x_\alpha := c(\alpha \mid X)$ satisfy the relations

$$x_\alpha = \sum_{\beta \in \mathbb{Z}^s} b_{2\alpha-\beta}x_\beta, \quad \alpha \in \mathbb{Z}^s.$$

Since $x_\alpha \geqslant 0$, $b_\alpha \geqslant 0$ and $\sum_{\beta \in \mathbb{Z}^s} b_{\alpha-2\beta} = \sum_{\beta \in \mathbb{Z}^s} x_\beta = 1$, the highest eigenvalue of the matrix $(b_{2\alpha-\beta}: \alpha, \beta \in \mathbb{Z}^s)$ is one. This eigenvalue is known to be simple among finitely supported sequences. Thus the solution of the above eigenvector equation will determine the necessary values of the cube spline. This leaves us to decide whether or not the Laurent polynomials $\psi((-1)^e z \mid X)$, $e \in E_s$, have common zeros in $(\mathbb{C}\setminus\{0\})^s$.

Below we record several examples of this computation for bivariate cube splines corresponding to the directions (1, 0), (0, 1), (1, 1), (−1, 1) repeated with multiplicities. We indicate the results of the computation by displaying the support of the cube spline and labeling the value of the spline at lattice points interior to its support. We have scaled the values so that they are all integers. The actual values of $c(\alpha \mid X)$ are obtained by dividing each integer by the sum of all the nonzero values of the cube spline at lattice points. The set of vectors which generate the

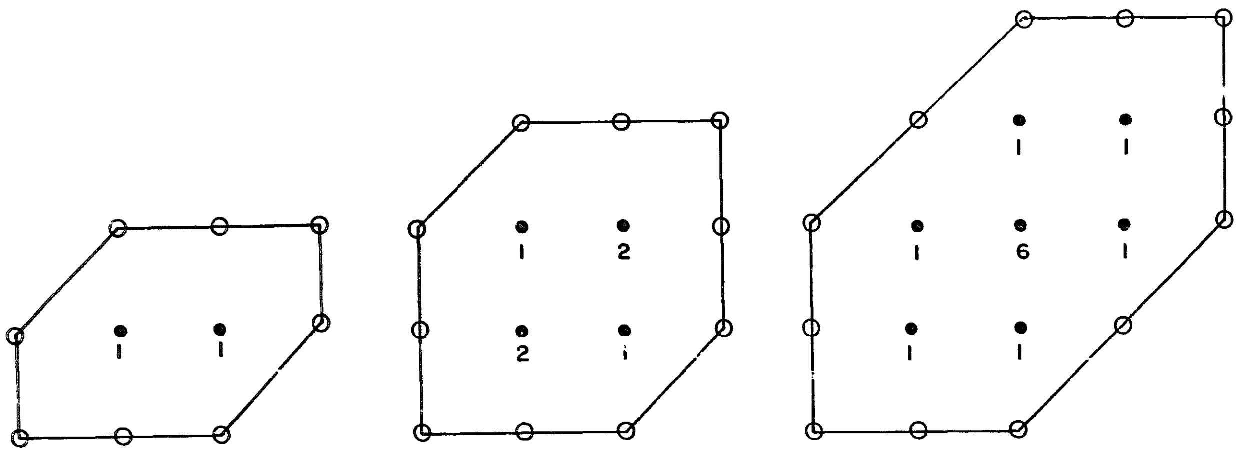

Fig. 3.1.

cube spline in each example can be identified by the shape of its support and so we do not specifically note this information.

Example 3.5. Let $X = X_{(m_1,m_2,m_3)}$ consist of vectors $\binom{1}{0}$ repeated m_1 times, $\binom{0}{1}$, m_2 times, and $\binom{1}{1}$, m_3 times, with m_1, m_2, $m_3 \geqslant 1$. Suppose $z = (x, y) \in (\mathbb{C}\setminus\{0\})^2$ is a common zero of $\psi(z(-1)^e)$, $e \in E_2 = \{(0,0), (0,1), (1,0), (1,1)\}$, where

$$\psi(z) = 2^{-m_1-m_2-m_3+2}(1+x)^{m_1}(1+y)^{m_2}(1+xy)^{m_3}\Phi(x, y),$$

$$\Phi(x, y) = \Phi\big(x, y \mid X_{(m_1,m_2,m_3)}\big).$$

We consider, four possiblities for (x, y).

Case 1. $x, y, xy \neq -1$ and $\Phi(\pm x, \pm y) = 0$.

Case 2. $x = -1$, $y \neq \pm 1$ and $\Phi(1, \pm y) = 0$.

Case 3. $x \neq \pm 1$, $y = -1$ and $\Phi(\pm x, 1) = 0$.

Case 4. $x, y \neq -1$, $xy = -1$ and $\Phi(x, x^{-1}) = \Phi(-x, -x^{-1}) = 0$.

With a systematic check of these four cases in each of the three cases of Fig. 3.1, we conclude no such common zero exists. Hence Theorem 3.4 establishes the existence of a fundamental function of the type (3.22) with $\phi(x) = c(x \mid X)$.

Example 3.6. In this example we add to the set $X_{(m_1,m_2,m_3)}$ the vector $\binom{-1}{1}$ repeated m_4 times, $m_4 \geqslant 1$. The new set $X = X_{(m_1,m_2,m_3,m_4)}$ leads to cube splines for which $\phi = c(\cdot \mid X)$ does not satisfy the conditions of Theorem 3.4. The reason for this is that the two vectors $z = (\pm \mathrm{i}, \mathrm{i})$ are common zeros of $b(z \mid X)$. Thus in this example we only checked for common zeros of $\Phi((-1)^e z \mid X)$, $e \in E_2$. For this purpose, we found it easier to use the polynomials $\Phi^e(z \mid X) = \sum_{\beta \in \mathbb{Z}^2} c(2\beta + e \mid X) z^\beta$ to check that in all cases there are no common zeros. Hence by Theorem 3.1 in each of the five cases of Fig. 3.2 there exists a fundamental function of the type (3.7).

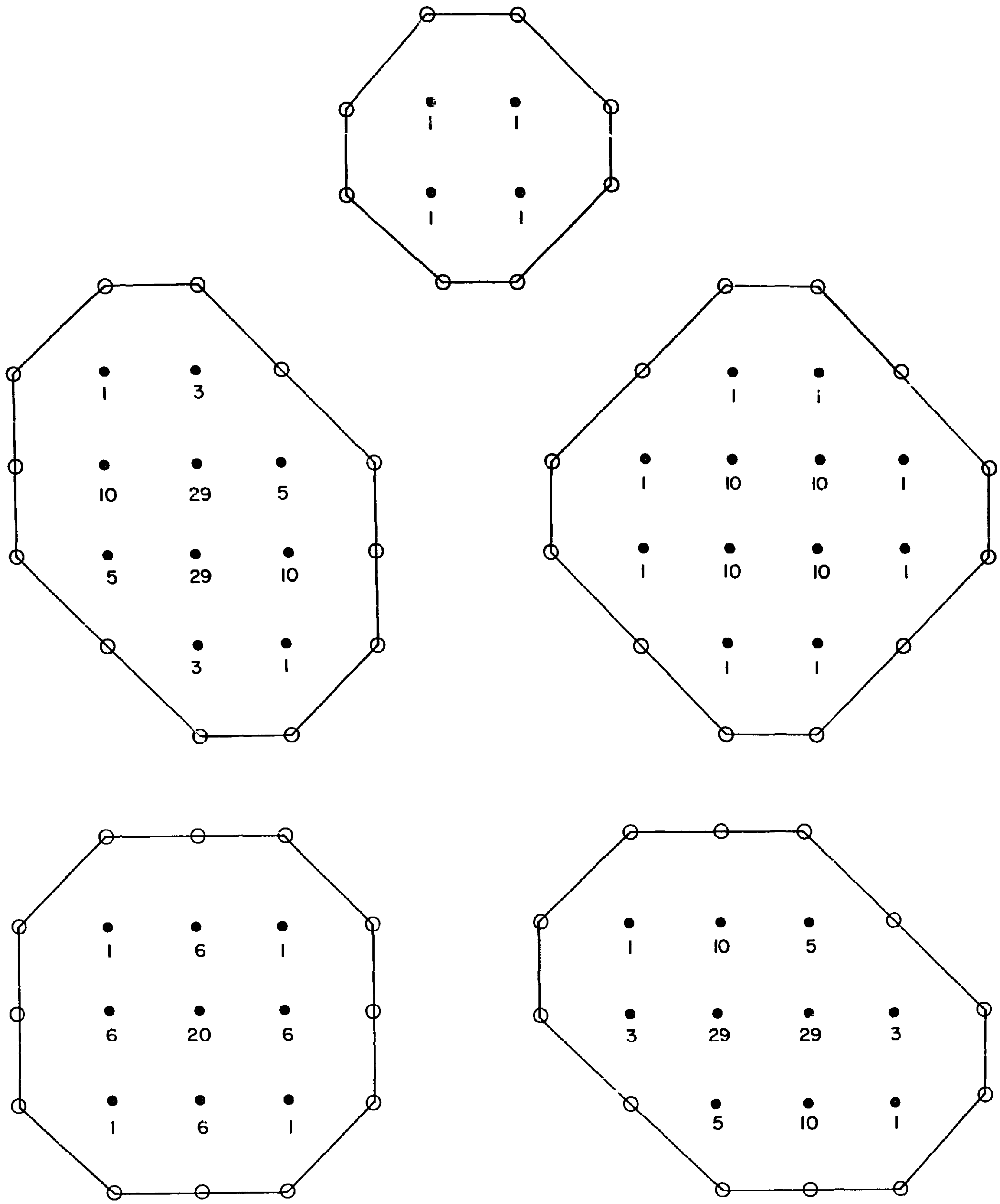

Fig. 3.2.

It would be interesting to have a general result about the construction of fundamental functions of the form (3.7) or (3.22) using cube splines.

We end this section with an application of Proposition 3.2 for univariate functions. Recall that a Laurent polynomial $b(z)$ is called a Hurwitz polynomial provided that all the zeros of $b(z)$ in $\mathbb{C}\setminus\{0\}$ have negative real parts. We will say b is a Pólya polynomial if it is a Hurwitz polynomial with real zeros. Alternatively,

$$b(z)=\sum_{j=m}^{n} b_j z^j, \quad b_m, b_n>0,$$

is a Pólya polynomial provided the bi-infinite Toeplitz matrix $(b_{i-j}: i, j\in\mathbb{Z})$ has all nonnegative minors (cf. [10, p.399]).

Corollary 3.7. *Let ϕ be a continuous function of compact support on $\mathbb{R}$ such that*

$$\sum_{j\in\mathbb{Z}} \phi(\tfrac{1}{2}j)z^j \tag{3.27}$$

is a Hurwitz polynomial. Then there exists a fundamental function of the type (3.19).

Proof. By multiplying (3.27) by a power of z we may assume without loss of generality that it is a polynomial which only has zeros in the left half plane. Suppose $\tau\in\mathbb{C}$ and $\tau=z^2$. Then

$$\sum_{j\in\mathbb{Z}} \phi(\tfrac{1}{2}j)z^j=\sum_{j\in\mathbb{Z}}\phi(j)\tau^j+z\sum_{j\in\mathbb{Z}}\phi(j+\tfrac{1}{2})\tau^j.$$

According to the Hermite–Biehler theorem (cf. [6, p.228]) the polynomials

$$\sum_{j\in\mathbb{Z}}\phi(j)z^j,\qquad \sum_{j\in\mathbb{Z}}\phi(j+\tfrac{1}{2})z^j$$

have real zeros which interlace each other. Hence, in particular, they have no common zeros. Thus the result follows from Theorem 3.4. □

Next, we recall the following result proven in [7]. Let

$$a(z)=\sum_{j=0}^{n+1} a_j z^j,\quad a_0, a_{n+1}>0, \tag{3.28}$$

be a Hurwitz polynomial. If $a(-1)=0$ and $a(1)=2$, then there is a unique continuous function ϕ which is positive on $(0, n+1)$, zero otherwise, such that

$$\phi(x)=\sum_{j=0}^{n+1} a_j\phi(2x-j),\quad x\in\mathbb{R}, \tag{3.29}$$

and

$$1=\sum_{j\in\mathbb{Z}}\phi(x-j),\quad x\in\mathbb{R}. \tag{3.30}$$

Moreover, the polynomial

$$\Phi(z)=\sum_{j\in\mathbb{Z}}\phi(j)z^j,\quad z\in\mathbb{C},$$

is a Pólya polynomial. We refer to ϕ as a ripplet.

Corollary 3.8. *Let* $a(z)$ *given by* (3.28) *be a Hurwitz polynomial such that* $a(-1)=0$, $a(1)=2$ *with corresponding ripplet* ϕ, *which satisfies* (3.29) *and* (3.30). *Then there is a fundamental function of the form* (3.19).

Proof. By our above remarks the polynomial

$$\sum_{j\in\mathbb{Z}} \phi(\tfrac{1}{2}j)z^j = a(z)\Phi(z), \quad z\in\mathbb{C}\setminus\{0\},$$

is a Hurwitz polynomial. Hence the result follows from Corollary 3.7. □

Let us refine these results. We will identify the unique fundamental function of minimal support of the form (3.7) where ϕ is a ripplet for which the interpolation operator I preserves polynomials of degree l ($\leqslant n$). To this end, we first observe that $\hat{\phi}(\pi)\neq 0$. To see this, we note that from the refinement equation (3.29) we have

$$2\hat{\phi}\left(\frac{\pi}{2^l}\right) = a\left(\exp\left(\frac{-\mathrm{i}\pi}{2^{l+1}}\right)\right)\hat{\phi}\left(\frac{\pi}{2^{l+1}}\right),$$

for $l=0, 1, 2,\ldots$. Since $\exp(-\mathrm{i}\pi/2^{l+1})$ is not in the left half plane for all $l\in\mathbb{Z}_+$, we see that if $\hat{\phi}(\pi)=0$, then $\hat{\phi}(\pi/2^l)=0$ for all $l=0, 1, 2,\ldots$. But this implies $\hat{\phi}(0)=0$, which contradicts (3.30).

Consequently, we conclude by Proposition 3.2 that if I preserves polynomials of degree l, the fundamental function

$$L(x) = \sum_{j\in\mathbb{Z}} d_j\phi(2x-j) \tag{3.31}$$

has the property that $d(-1)=\cdots=d^{(l)}(-1)=0$, $d(1)=2$, and $\hat{\phi}(2\pi j)=\cdots=\hat{\phi}^{(l)}(2\pi j)=0$, all $j\in\mathbb{Z}\setminus\{0\}$. Using the refinement equation (3.29) we see, again by (the proof of) Proposition 3.2 this latter fact is equivalent to $a(-1)=\cdots=a^{(l)}(-1)=0$ (see Remark 3.3, where d is replaced by a and L by ϕ). In summary, the fundamental function of Corollary 3.7 provides an interpolation operator which preserves polynomials of degree $\leqslant l$ if and only if both $d(z)$ and $a(z)$ have an $(l+1)$-fold zero at $z=-1$. Let us factor these polynomials as

$$d(z) = m^l(z)c(z), \tag{3.32}$$

$$m^l(z) = 2^{-l}(1+z)^{l+1} = \sum_{j=0}^{l+1} m_j^l z^j, \tag{3.33}$$

$$a(z) = m^l(z)b(z), \quad b(1)=1, \tag{3.34}$$

and introduce the function

$$\Gamma_l(x) := \sum_{j=0}^{l+1} m_j^l\phi(2x-j). \tag{3.35}$$

Then

$$L(x) = \sum_{j\in\mathbb{Z}} c_j\Gamma_l(x-\tfrac{1}{2}j) \tag{3.36}$$

and

$$\sum_{j\in\mathbb{Z}} \Gamma_l(\tfrac{1}{2}j)z^j = m^l(z)\Phi(z). \tag{3.37}$$

Although we will not make use of it, Γ_l satisfies the following functional equation:

$$\Gamma_l(\tfrac{1}{2}x) = \sum_{j\in\mathbb{Z}} g_j\Gamma_l(x-\tfrac{1}{2}j),$$

where

$$g(z) = m^l(z^2)b(z).$$

The equation (3.37) implies that $\{\Gamma_l(\frac{1}{2}j)\colon j\in\mathbb{Z}\}$ is a Pólya frequency sequence. Hence, Corollary 3.7 implies that there is a fundamental function of the type (3.36). By construction $\hat{\Gamma}_l^{(r)}(2\pi j)=0$, $r=0,1,\ldots,l$, for all $j\in\mathbb{Z}\setminus\{0\}$, and so it follows that the corresponding interpolation operator I, given by (3.2), is exact for polynomials of degree $\leqslant l$.

Let us identify the unique fundamental function of minimal support. We begin by observing that Γ_l has support in $[0, \frac{1}{2}(n+l)+1]$.

Case 1. $n+l=2r$. Then for any $1\leqslant m\leqslant 2r-1=n+l-1$ there exist unique polynomials

$$A(z)=\sum_{j=0}^{r-1} h_{2j}z^j,\qquad B(z)=\sum_{j=0}^{r-2} h_{2j+1}z^j,$$

such that

$$A(z)\sum_{j=0}^{r-1}\Gamma_l(j+1)z^j + B(z)\sum_{j=0}^{r}\Gamma_l(j+\tfrac{1}{2})z^j = z^{m-1}$$

(cf. [14, pp. 23–25]). We set $h_j=0$ for $j\notin\{0,1,\ldots,2r-2\}$ and define $c_j=h_{j+2m}$, $j\in\mathbb{Z}$. Thus we get

$$\Big(\sum_{j\in\mathbb{Z}} c_{2j}z^j\Big)\Big(\sum_{j\in\mathbb{Z}}\Gamma_l(j)z^j\Big)+\Big(\sum_{j\in\mathbb{Z}} c_{2j+1}z^j\Big)\Big(\sum_{j\in\mathbb{Z}}\Gamma_l(j-\tfrac{1}{2})z^j\Big)=1. \tag{3.38}$$

and so,

$$L(x)=\sum_{j=-2m}^{-2m+n+l-2} c_j\Gamma_l(x-\tfrac{1}{2}j)=\sum_{j=-2m}^{-2m+n+2l-1} d_j\phi(2x-j),\quad x\in\mathbb{R}. \tag{3.39}$$

Case 2. $n+l=2r-1$. In this case, we require $1\leqslant m\leqslant 2r-2=n+l-1$ and there exist unique polynomials

$$A(z)=\sum_{j=0}^{r-2} h_{2j}z^j,\qquad B(z)=\sum_{j=0}^{r-2} h_{2j+1}z^j,$$

such that

$$A(z)\sum_{j=0}^{r-1}\Gamma_l(j+1)z^j + B(z)\sum_{j=0}^{r-1}\Gamma_l(j+\tfrac{1}{2})z^j = z^{m-1}.$$

Again, we set $h_j=0$ for $j\notin\{0,1,\ldots,2r-3\}$ and define $c_j:=h_{j+2m}$, $j\in\mathbb{Z}$. This also leads to (3.38) and (3.39). We summarize these observations in the following theorem.

Theorem 3.9. *Suppose* $0 \leqslant l \leqslant n$, $1 \leqslant m \leqslant n+l-1$,

$$b(z) = \sum_{j=0}^{n-l} b_j z^j, \quad b(1) = 1,$$

is a Hurwitz polynomial. Set

$$a(z) = m^l(z) b(z),$$

and let ϕ *be the corresponding ripplet satisfying* (3.29) *and* (3.30). *Then there is a unique fundamental function with minimal support on* $[-m, -m+n+l]$ *of the form*

$$L(x) = \sum_{-2m}^{-2m+n+2l-1} d_j \phi(2x-j) = \sum_{-2m}^{-2m+n+l-2} c_j \Gamma_l(x - \tfrac{1}{2}j), \quad x \in \mathbb{R},$$

and the interpolation operator

$$(If)(x) = \sum_{j \in \mathbb{Z}} f(j) L(x-j), \quad x \in \mathbb{R},$$

is exact for $f \in \pi_l(\mathbb{R})$.

This theorem, in the case that $b(z) = 2^{-n+l}(1+z)^{n-l}$, is a consequence of results obtained in [3]. The approach there suggests an alternative derivation of Theorem 3.9. Namely, for every $k, j \in \mathbb{Z}$ we observe that

$$\Gamma_l(k + \tfrac{1}{2}j) = \sum_{r \in \mathbb{Z}} m^l_{2k-r} \phi(r-j).$$

Thus by the Cauchy–Binet formula the bi-infinite matrix $(\Gamma_l(k - \frac{1}{2}j))$, $k,j \in \mathbb{Z}$, has all nonnegative minors. Furthermore, one can then show by results in [7] that

$$\det \Gamma_l(k - \tfrac{1}{2}j) > 0,$$

$k = -m+1, \ldots, -m+n+l-1$, $j = -2m, \ldots, -2m+n+l-2$. From this fact we may solve the linear equation $L(j) = \delta_{0j}$, $j = -m+1, \ldots, -m+n+l+1$, to determine the coefficients $\{c_j \colon j \in \mathbb{Z}\}$ of L in (3.36).

Comments on Theorem 2.2

• Our original idea for the proof of part (ii) of Theorem 2.2 was based on the Quillen–Suslin theorem, cf. [11]. This result insures that if the matrix $A(z)$ is of full rank for $z \in (\mathbb{C} \setminus \{0\})^s$, it can be imbedded into a square matrix $M(z)$ with determinant one, whose first k rows agree with those of $A(z)$ and the remaining elements are Laurent polynomials. From $M(z)$, the matrix $C(z)$ is easily constructed as are the first l columns of $M(z)$. However, we realized only after reading the interesting paper [15] that the Nullstellensatz could also be used for its proof.
• The proof we presented is a "distillation" of the proof of Theorem 2.3 given in [15] "stripped" to its essential detail. Later, as we sought information on Gröbner bases techniques for polynomial ideal calculations we discovered [8], where the identical proof of Theorem 2.3 was given. The author of that paper also references [15].

References

[1] B. Buchberger, Gröbner bases: an algorithmic method in polynomial ideal theory, in: N.K. Bose, Ed., *Multidimensional System Theory* (Reidel, Dordrecht, 1985) 184–232.

[2] A.S. Cavaretta, W. Dahmen and C.A. Micchelli, *Stationary Subdivision*, Mem. Amer. Math. Soc. **93** (453) (Amer. Mathematical Soc., Providence, RI, 1991).

[3] W. Dahmen, T.N.T. Goodman and C.A. Micchelli, Compactly supported fundamental functions for spline interpolation, *Numer. Math.* **52** (1988) 639–664.

[4] W. Dahmen and C.A. Micchelli, Recent progress in multivariate splines, in: C.K. Chui, L.L. Schumaker and J.D. Ward, Eds., *Approximation Theory IV* (Academic Press, New York, 1983) 27–121.

[5] W. Dahmen and C.A. Micchelli, Subdivision algorithms for the generation of box spline surfaces, *Comput. Aided Geom. Design* **1** (2) (1984) 115–129.

[6] F.D. Gantmacher, *The Theory of Matrices, Vol. II* (Chelsea, New York, 1959).

[7] T.N.T. Goodman and C.A. Micchelli, On refinement equations determined by Pólya frequency sequences, *SIAM Math. Anal.* **23** (1992) 766–784.

[8] J.P. Guiver, The equation $Ax = b$ over the ring $\mathbb{C}[z, w]$, in: N.K. Bose, Ed., *Multidimensional System Theory* (Reidel, Dordrecht, 1985) 233–244.

[9] R.Q. Jia and C.A. Micchelli, On linear independence for integer translates of a finite number of functions, *Proc. Roy. Soc. Edinburgh Sect. A*, to appear.

[10] S. Karlin, *Total Positivity* (Stanford Univ. Press, Stanford, CA, 1969).

[11] T.Y. Lam, *Serre's Conjecture*, Lecture Notes in Math. **635** (Springer, Heidelberg, 1978).

[12] W. Rudin, *Functional Analysis* (McGraw-Hill, New York, 1973).

[13] B.L. van der Waerden, *Modern Algebra, Vol. II* (Frederick Ungar, New York, 1953).

[14] R.J. Walker, *Algebraic Curves* (Princeton Univ. Press, Princeton, NJ, 1950).

[15] D. Youla and G. Gnavi, Notes on n-dimensional system theory, *IEEE Trans. Circuits and Systems* **CAS-26** (2) (1979) 105–111.